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Generalized Cauchy Formulas and Numerical Differentiation*

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1. INTRODUCTION

In this paper, a general theory for numerical approximation formulas is developed which encompasses classical polynomial approximations as well as approximations involving transcendental functions. Assuming analyticity, it is shown that a large number of specific approximation formulas may be derived from the same basic complex contour integral, only one part of the integral needing adjustment to lead from one approximation formula to another. In particular, these formulas are developed for application to numerical differentiation and lead in a rather natural progression from Cauchy's formula for derivatives to a series for the differentiation of band-limited functions.

In the following section, the basic integral form is introduced and applied to derive several specific functional approximation formulas. This integral also serves as a representation for truncation error, and a general representation of error for derivatives is presented in Section 3. At the end of that section, we list a special case of that representation as the generalized Cauchy formula. In Section 4, that formula is applied and related to each of the corresponding functional approximation formulas from Section 2. General convergence questions are treated in the final section, and a modified form of the derivative expansion from the cardinal series is listed in closing.

2. THE BASIC REPRESENTATION

The integral form for the representation of the difference between an analytic function f and its approximation p, which serves as the basis for the

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following discussion, is

$$f(t)-p(t)=\frac{A(t)}{2\pi i}\int_C \frac{f(z)\,dz}{A(z)(z-t)}. \tag{1}$$

This form may be derived from the residue theorem, assuming C is a simple closed curve contained in a simply-connected domain in which f is analytic. The form of the approximation p depends on the zeros of what we call the "auxiliary" function, $A(z)$, which are interior to the curve. Concentrating on real approximations for simplicity, we assume that f is real on the real line and that t is a real value which must also be in the interior of the curve. For series representations where the contour must expand to enclose further values, we will give that indication by listing the contour as C_n.

In this section, a list of formulas derived from (1) will be given for a number of specific cases, some of which have been treated by various authors independently of any general theory. The authors in [7] state that one form of Lagrange's interpolation polynomial suggests the use of contour integration as in (1), since that form "characteristically resembles the sum of residue terms." They and the authors in [8, 14] concentrate on the use of (1) for approximations involving transcendental functions, while in [10] only polynomial approximations are considered. References to and applications of formulas related to (1) have occurred recently in other areas, as in [1, 5] for example. Methods for the numerical approximation of analytic functions that differ from that presented in this paper may be found in [4, 11, 12].

The following examples are numbered so as to correspond with applications of the generalized Cauchy formula in Section 4. Their sequence is intended to indicate a natural progression from one approximation formula to another.

2.1. *Taylor Series Approximation*

Let $A(z)=(z-t_0)^n$. This leads to an approximating polynomial of form $p(t)=\sum_{k=0}^{n-1} a_k(t-t_0)^k$, where

$$a_k=\frac{f^{(k)}(t_0)}{k!}=\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{A_k(z)} \qquad \text{with} \quad A_k(z)=(z-t_0)^{k+1}.$$

2.2. *Newton's Divided Difference Formula*

This time the zeros of $A(z)$ are to be distinct points, rather than having a multiple root at one point, so $A(z)=\prod_{i=0}^{n}(z-t_i)$.

The approximation is of form $p(t)=a_0+\sum_{k=1}^{n} a_k \prod_{j=0}^{k-1}(t-t_j)$, where

$$a_k=f[t_0,\ldots,t_k]=\frac{1}{2\pi i}\int \frac{f(z)\,dz}{A_k(z)},$$

with $A_k(z)=\prod_{i=0}^{k}(z-t_i)$.

One special case of this approximation is important for our subsequent generalizations to transcendental auxiliary functions, and that is the case where the data points are equally spaced. Keeping the above form, but normalizing the points so that they are symmetric about the origin, we may write the auxiliary function in the following suggestive form,

$$A(z) = C(n, h)\, z \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2 h^2}\right),$$

where h is the common (real) step length, and $C(n, h) = (-1)^n\,(n!\,h^n)^2$.

2.3. *Hermite Interpolation*

This corresponds to a combination of the first two methods, since if powers are allowed in the individual factors, then the approximation agrees with a prescribed number of functional *derivatives* as well as with functional values. The number of matching derivatives may be the same for each data point, or may be arranged to differ at each value. For the latter situation, the varying powers give the auxiliary function the form

$$A(z) = \prod_{i=0}^{n} (z - t_i)^{n_i}.$$

Now the approximation matches $f^{(k)}(t)$ at $t = t_i$ for $0 \leqslant k \leqslant n_i - 1$.

If $A_j(t) = A(t)/(t - t_j)^{n_j}$, then the form of the approximation according to [10, p. 49] is $p(t) = \sum_{j=0}^{n} \sum_{i=0}^{n_j-1} (f^{(i)}(t_j)/i!)\ A_j(t)\ \sum_{s=0}^{n_j-1-i} c_s^{(j)}(t - t_j)^{i+s}$, where the final sum is the first part of the power series of $1/A_j(z)$ about t_j.

2.4. *Walsh's Rational Approximation* [15, *p.* 186]

The auxiliary function is now made into a rational function itself. In its denominator is placed a product with roots at a set of points $\{\alpha_i\}$, which may lie inside the contour of integration, but are assumed to be distinct from its zeros. It then has the form

$$A(z) = \prod_{i=0}^{n} (z - \beta_i) \Big/ \prod_{i=1}^{n} (z - \alpha_i).$$

Through Eq. (1), this constructs the rational approximation function which agrees with $f(t)$ at each $t = \beta_i$ point and has poles at each $t = \alpha_i$.

We now move to transcendental auxiliary functions.

2.5. *The Whittaker–Shannon Interpolation Formula* (*the Cardinal Series*)

The auxiliary function with zeros at all integral multiples of a fixed real step length h is $A(z) = \sin(\pi z/h)$.

If the contour of integration is arranged to be symmetric about the origin so as to contain $2n+1$ of the data points in its interior, the resulting approximation formula is

$$p(t) = \sum_{k=-n}^{n} f(kh) \frac{\sin\{\pi(t-kh)/h\}}{\pi(t-kh)/h}. \tag{2}$$

Thus the resulting approximation function is also transcendental.

A translation of this auxiliary function allows for an approximation based on an arbitrary set of *equally spaced* data points. Recall that the infinite product expansion for this auxiliary function is

$$\sin\left(\frac{\pi z}{h}\right) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 h^2}\right) \tag{3}$$

From (3), the interpolation formula (2) may be seen as a generalization of the special Newton's formula in Example 2.2.

2.6. *A Bessel Function Approximation Formula*

This is an example of a transcendental auxiliary function whose zeros are not equally spaced. If $A(z) = J_0(z)$, then the data points are placed at the zeros of $J_0(z)$, $\{j_{0,k}\}$. These values have a representation in series form, but have already been tabulated to several decimal places in [13]. The resulting approximation has the form

$$p(t) = \sum_{k=-n}^{n} f(j_{0,k}) \frac{J_0(t)}{J_1(j_{0,k})(j_{0,k} - t)},$$

where the contour of integration is assumed to be symmetric about the origin.

If the function f is a Hankel transform, this sum assumes a special form listed in [8], where it has been shown to then converge to f.

2.7. *Approximations Involving Arbitrarily Chosen Data Points*

Suppose that the set of points $\{t_k\}$ is to be used as data points, where $|t_1| < |t_2| < |t_3| < \cdots$. Under the conditions that $|t_n| \to \infty$ as $n \to \infty$ and $\sum_{k=1}^{\infty} 1/|t_k| < \infty$, the Weierstrass canonical product $P(z) = \prod_{k=1}^{\infty} (1 - z/t_k)$ provides an entire function which has a simple zero at each data point with no other zeros in the plane.

If the auxiliary function is chosen to be this product, the resulting approximation with error given by (1) has the familiar form

$$p(t) = \sum_{k=0}^{n} f(t_k) \frac{A(t)}{(t-t_k) A'(t_k)}.$$

3. The Integral Representation for Derivatives

Let $E(t)=f(t)-p(t)$. With the conditions on $f(z)$ and $A(z)$ as stated in Section 2, we have

THEOREM.

$$E^{(n)}(t)=\frac{n!}{2\pi i}\int_C \frac{f(z)\sum_{k=0}^{n}\{A^{(k)}(t)/k!\}(z-t)^k\,dz}{A(z)(z-t)^{n+1}}. \tag{4}$$

Proof. If $n=1$, the continuity of $f(z)/\{A(z)(z-t)\}$ on C allows differentiation to be taken inside the integral sign in (1), establishing the result in that case.

Now suppose that (4) holds for some $n>1$. Differentiating under the integral sign is again valid, so that with $P_n(z,t)=\sum_{k=0}^{n}\{A^{(k)}(t)/k!\}(z-t)^k$, we find

$$E^{(n+1)}(t)=\frac{n!}{2\pi i}\int_C \frac{f(z)}{A(z)}\left\{\frac{(n+1)}{(z-t)^{n+2}}P_n(z,t)+\frac{1}{(z-t)^{n+1}}\frac{d}{dt}P_n(z,t)\right\}dz.$$

Since the derivative with respect to t of $P_n(z,t)$ equals $A^{(n+1)}(t)(z-t)^n/n!$, we obtain

$$\begin{aligned}E^{(n+1)}(t)&=\frac{n!}{2\pi i}\int_C \frac{f(z)}{A(z)(z-t)^{n+2}}\left\{(n+1)\,P_n(z,t)+\frac{(z-t)\,A^{(n+1)}(t)(z-t)^n}{n!}\right\}dz\\&=\frac{(n+1)!}{2\pi i}\int_C \frac{f(z)}{A(z)(z-t)^{n+2}}P_{n+1}(z,t)\,dz,\end{aligned}$$

completing the proof.

COROLLARY. *If L is a linear differential operator of order n with constant coefficients, then*

$$L(D)\,E(t)=\frac{1}{2\pi i}\int_C \frac{f(z)\sum_{k=0}^{n}\{L^{(n-k)}(D)\,A(t)\}(z-t)^k}{A(z)(z-t)^{n+1}}\,dz. \tag{5}$$

Proof. If $n=0$, the formula follows from the previous theorem, so suppose that the corollary is true for operators of order n. Let L be an operator with order $n+1$, and characteristic equation $\sum_{k=0}^{n+1} a_k z^k$. If L_n is the operator of order n with characteristic equation equal to all but the last term of the one for L, then $L_n(D)\,E(t)$ has form (5) by the induction hypothesis.

By the previous theorem, $a_{n+1} D^{n+1} E(t)$ has form (4) with the sum

$$\sum_{k=0}^{n+1} \{a_{n+1}((n+1)!/k!)\,D^k A(t)\}(z-t)^k$$

multiplying the factor $f(z)/\{A(z)(z-t)^{n+2}\}$ in the integrand. Summing over a common denominator and assuming the factor above as a multiple, the relation $L(D)\,E(t) = L_n(D)\,E(t) + a_{n+1} D^{n+1} E(t)$ gives the rest of the integrand for $L(D)\,E(t)$ as

$$a_{n+1}(n+1)!\,A(t) + \sum_{k=1}^{n+1} \left\{ L_n^{(n+1-k)}(D)\,A(t) + a_{n+1} \frac{(n+1)!}{k!} D^k A(t) \right\} (z-t)^k.$$

But $L^{(n+1-k)}(z) = L_n^{(n+1-k)}(z) + (n+1)!\,a_{n+1} z^k/k!$, so this reduces to $\sum_{k=0}^{n+1} \{L^{(n+1-k)}(D)\,A(t)\}(z-t)^k$, completing the proof.

From the corollary, if the function $A(z)$ were actually zero-free, then the error expression $E(t)$ is simply $f(t)$. Thus (5) gives an integral representation for the linear differential operator $L(D) f(t)$. For example, if $A(z) = \exp(bz)$ and $L(D) f = a_0 f + a_2 f''$, we find that

$$a_0 f(t) + a_2 f''(t) = \frac{\exp(bt)}{2\pi i} \int_C \frac{f(z)\{2a_2 + 2a_2 b(z-t) + (a_0 + a_2 b^2)(z-t)^2\}}{\exp(bz)(z-t)^3}\,dz.$$

Note that the general form of such a representation is not dependent on the function $f(z)$, but changes with the operator L and the function $A(z)$, since it is the derivatives of L applied to $A(t)$ that appear in (5).

If $L(z) = z^n$ and $A(z)$ is a non-zero constant, then (5) reduces to Cauchy's formula for derivatives. In fact, if $A(z)$ were not necessarily zero-free but had the property at some $z = j$ that $A(j) \neq 0$ but that $L^{(k)}(D)\,A(j) = 0$ for $k = 0, \ldots, n-1$, then (5) has the following form similar to Cauchy's:

$$L(D)\,E(j) = \frac{n!\,a_n A(j)}{2\pi i} \int_C \frac{f(z)}{A(z)(z-j)^{n+1}}\,dz.$$

The result we wish to emphasize, however, comes directly from the theorem. Recall first that the approximation formulas from Section 2 are based on points which are the zeros of the auxiliary function. If $A(z)$ actually has a zero of order n at $z=j$, then (4) becomes

$$E^{(n)}(j)=\frac{A^{(n)}(j)}{2\pi i}\int_C \frac{f(z)}{A(z)(z-j)}\,dz. \tag{6}$$

Thus, at any data point, we obtain an analog of (1) for differentiation. We modify this form slightly, using $A_j(z)=A(z)/(z-j)^n$ so that $A_j(j)=A^{(n)}(j)/n!$. Using this in the above, we obtain what we call the generalized Cauchy formula:

$$E^{(n)}(j)=\frac{n!\,A_j(j)}{2\pi i}\int_C \frac{f(z)\,dz}{A_j(z)(z-j)^{n+1}}. \tag{7}$$

In the next section, we will show how the approximation formulas of Section 2 can be used with this form to obtain specific generalizations of the standard Cauchy formula.

4. Applications

In this section, we concentrate attention on the function $A_j(z)$ and its forms as we apply the generalized Cauchy formula (7) to the approximation formulas from Section 2. Using $t=j$ as a data point and $A_j(z)$ as defined in Section 3, the forms of this function progress from constants to transcendental functions and lead from the classical Cauchy formula for derivatives to derivative approximation formulas involving transcendental functions. In the following, we refer to $A_j(z)$ as the Auxiliary function for Derivative Approximation, or simply ADA.

4.1. First, $A_j(z)=1$. This is derived from the auxiliary function for Taylor series, $A(z)=(z-j)^n$. Since the Taylor polynomial was of degree $n-1$ and the generalized Cauchy formula applied to the derivative of degree n, $E^{(n)}(j)=f^{(n)}(j)$. For this case, (7) becomes the classical Cauchy formula for derivatives.

4.2. Next, the ADA is generalized to be a polynomial, but with distinct roots:

$$A_j(z)=\prod_{\substack{i\neq j\\ i=0}}^{n}(z-t_i).$$

In this instance, this function was developed from Newton's formula, where

$A(z) = \prod_{i=0}^{n} (z - t_i)$. The generalized Cauchy formula applies this time to the first derivative:

$$f'(t_j) - p'(t_j) = \frac{\prod_{i \neq j} (t_j - t_i)}{2\pi i} \int_C \frac{f(z)\, dz}{\{\prod_{i \neq j} (z - t_i)\}(z - t_j)^2}$$
$$= \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{L_j(z)(z - t_j)^2},$$

where $L_j(z)$ is the standard Lagrangian coefficient function.

In particular, we consider the form of the special case when the data points are equally spaced at the points $t = ih$, $0 \leqslant i \leqslant n$. Evaluating when i equals zero, as is standard practice, gives

$$f'(0) - \left\{\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - + \cdots + \frac{(-1)^n \Delta^{n-1}}{(n-1)}\right\} f(0)$$
$$= \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{\Pi_{k=1}^{n} (1 - z/(kh))\, z^2}. \tag{8}$$

This form has implications for the transcendental case to follow, both from the form of the approximation and from the form of the denominator in the integrand.

4.3. Allowing powers in the individual factors provides for the approximation of derivatives of higher orders. This corresponds to Hermite interpolation, and the ADA is

$$A_j(z) = \prod_{i \neq j} (z - t_i)^{n_i}.$$

Where the Hermite approximation function has derivatives which match $f^{(k)}(t)$ for $0 \leqslant k \leqslant n_j - 1$, the generalized Cauchy formula yields information about the n_jth derivative:

$$f^{(n_j)}(t_j) - p^{(n_j)}(t_j) = \frac{n_j! \prod_{i \neq j} (t_j - t_i)^{n_i}}{2\pi i}$$
$$\times \int_C \frac{f(z)\, dz}{\{\prod_{i \neq j} (z - t_i)^{n_i}\}(z - t_j)^{n_j + 1}}.$$

4.4. Continuing to use $\{t_i\}$, $0 \leqslant i \leqslant n$, as the set of zeros, and allowing division, we obtain $A_j(z) = \prod_{i \neq j} (z - t_i) / \prod_{i=1}^{n} (z - \alpha_i)$, which relates to Walsh's rational approximation. The generalized Cauchy formula applies to $f'(t_j) - p'(t_j)$ in this case and is only a minor modification of the form from Section 4.2. It modifies that form by the multiple $\prod_{i=1}^{n}(z-\alpha_i)/\prod_{i=1}^{n}(t_j-\alpha_i)$.

4.5. Next, we consider the case where the ADA is transcendental with simple zeros at all the integers except the point of approximation, and first use $A_j(z)=\sin \pi z/(z-j)$. We shall return to this case in Section 5 with a modification of the above. This function, however, relates to the usual cardinal series (2) and yields the following example of the generalized Cauchy formula:

$$f'(j)-\sum_{\substack{i\neq j\\ i=-n}}^{n} f(i)\frac{(-1)^{j-i}}{j-i}$$
$$=\frac{\pi\cos(\pi j)}{2\pi i}\int_{C_n}\frac{f(z)\,dz}{\{\sin(\pi z)/(z-j)\}(z-j)^2}. \tag{9}$$

Note first the similarities between this case and that of approximation by polynomials using equally spaced data points (8). In each case, the series is not absolutely convergent, and the forms for each resemble the alternating harmonic series. The infinite product representation for $\sin \pi z$, (3), also appears as a generalization of the function used in the denominator of the integrand of (8).

4.6. *Derivative Approximation Using Bessel Functions*

Writing j_k instead of $j_{0,k}$, since the data points are assumed to be the zeros of $J_0(z)$, the ADA is $A_m(z)=J_0(z)/(z-j_m)$. Thus (7) has the form

$$f'(j_m)-\frac{1}{2j_m}f(j_m)-\sum_{\substack{k\neq m\\ k=-n}}^{n} f(j_k)\frac{J_1(j_m)}{J_1(j_k)(j_m-j_k)}$$
$$=\frac{J_1(j_m)}{2\pi i}\int_{C_n}\frac{f(z)\,dz}{\{J_0(z)/(j_m-z)\}(z-j_m)^2}.$$

For computation purposes, the irregular spacing of the data points for this case should not cause difficulty. Formulas are well known for the zeros of $J_0(z)$ as well as the values of $J_1(z)$ at those zeros. The first 150 of those values are listed in [13], for example.

Excluding the behavior of $f(j_k)$, note that the sum above is again not absolutely convergent, as the values $J_1(j_k)$ actually decrease as k increases.

4.7. The generalized Cauchy formula provides the form of the error in derivative approximation and is listed, (7), in the most general case. Corresponding to Example 2.7, the approximation formula that results from the initial auxiliary function having simple zeros at points $\{t_k\}$ is of the form

$$p'(t_j) = f(t_j)\frac{A''(t_j)}{2A'(t_j)} + \sum_{k \neq j} f(t_k)\frac{A'(t_j)}{A'(t_k)(t_j - t_k)}.$$

5. Convergence Questions

The similarity of the integral parts of the error forms, the one for differentiation (6) being precisely like that for the regular function appoximation case (1), means that convergence for both kinds of approximation may be treated simultaneously.

The point made in previous sections that the cardinal series may be seen as the transcendental generalization of polynomial approximation at equally spaced points also appears in terms of similarity of convergence criterion. Where the data points increase without bound, Stirling's approximation, which uses data symmetric about the origin, cannot converge unless the function $f(z)$ is entire of exponential type [6, p. 157]. Further, if $|f(re^{i\theta})| < Me^{ar}$, then the series will converge everywhere if $a < 2\log(1+\sqrt{2})\, h$, but will diverge everywhere except where it terminates if $a > \pi/h$. Since the auxiliary function for the cardinal series is $A(z) = \sin(\pi z/h)$, the integral form (1) can be used to show that the band-limited functions for which this series will converge are also of exponential type $a < \pi/h$.

As stated above, a similar convergence criterion follows immediately for the derivative approximation (9). Error estimates using Fourier transforms for that same approximation formula were discussed in [3]. Kramer, in [9], studied this approximation for band-limited functions and stated: "In tests it has been found that when the series is truncated at $n = 15$, the error is less than 0.1 per cent." However, Kramer did not state for which functions or for which values these tests had been done, nor did he give a method to determine error. But, as is emphasized in this article, we may take a detailed analysis already completed by Jagerman and Fogel [7] for the convergence of the cardinal series in the general case and apply that estimate to this derivative approximation.

If f is entire of exponential type a, where $a < \pi/h$, and $f \in L^2(R)$, i.e., f is a band-limited function, then $A(y) = \max_{-\infty < x < \infty} |f(x+iy)| \leqslant (K/|y|)\, e^{\pi|y|/h}$ as $|y| \to \infty$, for some constant K. Jagerman and Fogel choose the contour of integration for the error integral (1) as a square of side length $2n+1$ centered at the origin, and under our assumptions it would have sides parallel to the coordinate axes . The integral over the upper horizontal line, $I_{n,1}$, is bounded in the following manner:

$$|I_{n,1}| \leqslant \frac{4K_1K}{n+\frac{1}{2}}, \text{ if } K_1 \geqslant \frac{(2n+1)h}{\pi\{(2n+1)h - 2|t|\}} \qquad \text{for all } n,$$

assuming n is large enough so that $\exp\{\pi(2n+1)/h\} \geqslant 2$. A similar result holds for the other horizontal side. If $I_{n,4}$ is the integral over the right vertical side, they show that

$$|I_{n,4}| \leqslant \frac{1}{\pi} \int_0^\infty A(y)/D(n, y, t)\, dy, \tag{10}$$

where

$$D(n, y, t) = (\sqrt{(n+\tfrac{1}{2})^2 + y^2} - |t|) \cosh(\pi y/h). \tag{11}$$

The resulting bound that would apply to the total error integral for this general case is then

$$|I| \leqslant \frac{8K_1K}{n+\frac{1}{2}} + \frac{2}{\pi} \int_0^\eta \frac{A(y)}{D(n, y, t)}\, dy + \frac{4K}{\pi} \int_\eta^\infty \frac{1}{D(n, y, t)}\, dy,$$

where $\eta > 0$ is arbitrary. This expression vanishes as $n \to \infty$ and does provide an estimate of the error for any fixed n.

Perhaps this estimate could be improved. Another approach is to modify the auxiliary function used in Example 4.5 so as to obtain more rapid convergence of the series, although this approach may require the integral remainder to again be estimated.

If $E(z)$ is a zero-free entire function, then the product $E(z) \sin(\pi z)$ has the same zeros as before. If this product is used as a new auxiliary function, then the data points remain at the integers and the generalized Cauchy formulas involving (2) and (9) are modified to have the following forms:

$$f(t) - \sum_{k=-n}^{n} f(k) \frac{E(t) \sin \pi(t-k)}{E(k)\, \pi(t-k)}$$
$$= \frac{E(t) \sin(\pi t)}{2\pi i} \int_{C_n} \frac{f(z)\, dz}{E(z) \sin \pi z (z-t)} \tag{12}$$

and

$$f'(j) - E(j) \sum_{\substack{k \neq j \\ k=-n}}^{n} f(k) \frac{(-1)^{j-k}}{E(k)(j-k)} - f(j) \frac{E'(j)}{E(j)}$$
$$= \frac{E(j)\, \pi(-1)^j}{2\pi i} \int_{C_n} \frac{f(z)\, dz}{E(z) \sin \pi z (z-j)}. \tag{13}$$

If $E(k)$ is of large asymptotic growth as $|k| \to \infty$, then each of the series will converge more rapidly than before. Choices for $E(z)$ include all functions of the form $\exp(h(z))$, where $h(z)$ is entire. However, for any choice, estimates must be made on the integral remainder to determine for which class of functions $f(z)$ that remainder becomes small in the limit.

As an example, suppose $E(z) = \exp(z^2)$. If the function $f(z)$ is translated so as to be evaluated at $j = 0$, then (13) has the form

$$f'(0) - \sum_{\substack{k \neq 0 \\ k=-n}}^{n} f(k) \frac{(-1)^{k+1}}{k \exp(k^2)} = \frac{1}{2i} \int_{C_n} \frac{f(z)\,dz}{z \sin \pi z \exp(z^2)}. \tag{14}$$

The above series will now converge extremely rapidly, although the remainder integral may not vanish completely. However, for band-limited functions whose growth lags behind that of $\sin \pi z$, this remainder will be shown to be quite small.

Suppose that $f(z)$ is a band-limited function with exponential type $\tau < \pi$. It follows from Theorem 6.2.6 [2, p. 83], that $|f(x+iy)| \leqslant Me^{\tau|y|}$, where $M = \max |f(x)|$. Since $1/|\sin \pi z| \leqslant 4e^{-\pi|y|}$ for $y \geqslant \ln 2/(2\pi)$, we have $|f(z)/\sin \pi z| \leqslant 4Me^{-A|y|}$, where $A = \pi - \tau$.

Let I' be the remainder integral in (14) above. We follow an analysis similar to the one presented previously, but modify the contour of integration to be a rectangle centered at the origin with width $2n+1$ and height A. If $I'_{n,1}$ is the integral over the upper horizontal line, then

$$|I'_{n,1}| \leqslant \frac{1}{2} \max \left| \frac{f(z)}{z \sin \pi z} \right| \int_{-(n+1/2)}^{n+1/2} e^{-x^2 + A^2/4}\,dx,$$

where the maximum is taken for $z = x + iA/2$, and $|x| \leqslant n + \frac{1}{2}$. Using the estimate above for these values,

$$\max \left| \frac{f(z)}{z \sin \pi z} \right| \leqslant \frac{2}{A} 4Me^{-A(A/2)} = \frac{8M}{A} e^{-A^2/2},$$

so

$$|I'_{n,1}| \leqslant \frac{8M}{A} e^{-A^2/2 + A^2/4} \sqrt{\pi}\, \operatorname{erf}\left(n + \frac{1}{2}\right).$$

Since $\operatorname{erf}(n+1/2) \leqslant 1$, this has the form $|I'_{n,1}| \leqslant (8M/A)\, e^{-A^2}/4$, and the estimate for the integral over the lower horizontal side would be similar.

If $I'_{n,4}$ is the integral over the right vertical side, then $|I'_{n,4}| \leqslant e^{A^2/4 - (n+1/2)^2} |I_{n,4}|$, where $|I_{n,4}|$ is as in (10). Thus the integrals over the vertical sides approach zero rapidly. Neglecting those parts, the total remainder integral may be bounded by $(16M/A)\, e^{-A^2/4}$. This is an estimate for the maximum error, so the actual error may be considerably smaller. In any case, if A is fairly large, this estimate gives a small value for the remainder.

For step lengths h other than $h=1$, this result is modified in that $A=\pi/h-\tau$. If h is small, then τ can be relatively large while retaining a large value for A. The data points would be at $t=kh$, for integers k, and the new term in the denominator of the series would be $\exp(k^2h^2)$. For small h, the convergence of the series would not be as rapid as when $h=1$, but it is still much more rapid than when unmodified; since A could be large, formula (14) would be applicable to a wide class of functions.

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